Lecture 22

Today we will cover a bunch of continuous distributions very quickly and with little detail.

The Gamma Distribution

We say that X has the gamma distribution with parameters (α, λ), $\alpha > 0$, $\lambda > 0$ if X has a density function given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Here $\Gamma(\alpha)$ is the gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} \, dy.$$

Exercise: Use integration by parts to show that

$$\Gamma(\alpha) = (\alpha - 1) \Gamma(\alpha - 1).$$

It follows from the exercise that, for an integer $n \geq 0$.

$$\Gamma(n) = (n-1)!$$

Therefore the Γ-distribution with parameters (α, n) has pdf given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

This should remind you of the exponential distribution, and indeed this is a generalization:

- Suppose some event is distributed as a Poisson process on an interval $[0, \infty)$ with average λ.
- If X is the time until the first event occurs then X is exponential with parameter λ.
- If X is the time until *n events occur*, (e.g. how long do I need to wait for 10 earthquakes to happen?) then X has a Γ-distribution with parameters (n, λ).

Expectation and Variance

Let X be Γ with params (α, λ). Then

$$E[X] = \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda x e^{-\lambda x} (\lambda x)^{\alpha-1} dx$$

$$= \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty \underline{\lambda}\, e^{-\lambda x} (\lambda x)^{\alpha} \underline{dx}$$

sub $t = \lambda x$
$dt = \lambda dx$

$$= \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha} dt = \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)}$$

$$= \frac{\alpha}{\lambda}.$$

Exercise: $Var(X) = \alpha/\lambda^2$

The Weibull Distribution.

- Commonly used in engineering
- Consider an object (computer, car, person) that experiences death (failure) when any of its parts fail (logic board, alternator, heart).
- then a Weibull distribution provides a close approximation to the lifetime of the object.
- A random variable X is said to have a Weibull distribution with parameters ν, α, β if

X has a cumulative distribution function given by

$$F_X(x) = \begin{cases} 0 & \text{if } x \le \nu \\ 1 - \exp\left(-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right) & x > \nu \end{cases}$$

- Note that if X is Weibull with $\nu = 0$, $\beta = 1$, then X is exponential with parameter $\lambda = 1/\alpha$
- Differentiating, we have

$$f_X(x) = \begin{cases} 0 & x \le \nu \\ \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right) & x > \nu. \end{cases}$$

For the expected value

$$E[X] = \int_{\nu}^{\infty} x \, \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right) dx$$

$$u = \left(\frac{x-\nu}{\alpha}\right)^{\beta}$$

$$du = \beta\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} dx$$

$$x = \alpha u^{1/\beta} + \nu$$

$$= \int_0^{\infty} \left(\frac{\alpha u^{1/\beta} + \nu}{\alpha}\right) u \, e^{-u} \, du$$

$$= \int_0^{\infty} u^{\frac{\beta+1}{\beta}} e^{-u} + \frac{\nu}{\alpha} u \, e^{-u} \, du.$$

$$= \Gamma\left(\frac{\beta+1}{\beta} + 1\right) + \frac{\nu}{\alpha}\Gamma(2)$$

$$= \Gamma\left(\frac{2\beta+1}{\beta}\right) + \frac{2\nu}{\alpha}$$

Exercise: Var(X).

Cauchy Distribution

A random variable X is said to have Cauchy distribution with parameter θ, $-\infty < \infty < \infty$ if X has pdf

$$f_X(x) = \frac{1}{\pi(1+(x-\theta)^2)}$$

See example 5.6.3.6b in the textbook

Exercise: Can you find the mean $E[X]$?

The Beta Distribution

A Random variable X is said to have a beta distribution if it has density function

$$f_X(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1. \\ 0 & \text{otherwise.} \end{cases}$$

Here $B(a,b)$ (B is capital β, not capital b) is the Beta function,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

When $a=b=1$, the Beta distribution is uniform $(0,1)$.

Fact: $B(a,b)$ satisfies the following identity:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

From this fact, it follows that

$$\frac{B(a+1,b)}{B(a,b)} = \left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}\right)\left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right) = \frac{a}{a+b}$$

since $\Gamma(x+1) = x\Gamma(x)$. From here, we can compute the expected value, since

$$E[X] = \frac{1}{B(a,b)} \int_0^1 x \, x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)} \int_0^1 x^a (1-x)^{b-1}\,dx = \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{a}{a+b}.$$

Similarly: $$E[X^2] = \frac{1}{B(a,b)} \int_0^1 x^{a+1}(1-x)^{b-1}\,dt$$

$$= \frac{B(a+2,b)}{B(a,b)} \underbrace{\left(\frac{B(a+1,b)}{B(a+1,b)}\right)}_{=1.}$$

$$= \frac{B(a+2,b)}{B(a+1,b)} \cdot \frac{B(a+1,b)}{B(a,b)}$$

$$= \left(\frac{a+1}{a+b+1}\right)\left(\frac{a}{a+b}\right)$$

and so $$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b)^2 + a + b} - \frac{a^2}{(a+b)^2}$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$

The Pareto Distribution

Suppose that X is an exponential RV with parameter λ. We say that a RV Y is Pareto with parameters $a>0$ and λ if

$$Y = ae^X.$$

For $y \geq 0$, we have

$$\begin{aligned} P(Y>y) &= P(ae^X > y) \\ &= P(e^X > y/a) \\ &= P(X > \log(y/a)) \\ &= e^{-\lambda \log(y/a)} \qquad \text{(since } X \text{ is exponential)} \\ &= e^{\log((y/a)^{-\lambda})} \\ &= \left(\frac{y}{a}\right)^{-\lambda} = \left(\frac{a}{y}\right)^{\lambda} \end{aligned}$$

Hence $F_Y(y) = P(Y \leq y) = 1 - P(Y>y) = 1 - a^\lambda y^{-\lambda}$ (for $y \geq a$).

Differentiating gives a pdf:

$$f_Y(y) = \lambda a^\lambda y^{-(\lambda+1)} \quad \text{for } y \geq a.$$